

# An irreducible symplectic orbifold of dimension 6 with a Lagrangian Prym fibration

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## Abstract

A new example of an irreducible symplectic variety of dimension 6, with only finite quotient singularities, is described as a relative compactified Prymian of a family of genus 4 curves with involution. It is associated to a K3 surface which is a double cover of a cubic surface. It has a natural Lagrangian fibration in abelian 3-folds with polarization type  $(1,1,2)$ . It does not admit any symplectic resolution.

## Introduction

An irreducible symplectic variety is a simply connected compact Kähler manifold which has a unique holomorphic symplectic structure. This notion can be considered as a generalization to higher dimensions of a K3 surface, which is equivalently an irreducible symplectic surface.

Very few examples of irreducible symplectic varieties are known, up to deformation equivalence. All of them come in the setting of moduli spaces of sheaves on K3 or abelian surfaces. Such a moduli space  $\mathcal{M}$ , indeed, inherits the symplectic structure on its smooth locus from the surface, as Mukai showed in [Mu]. Moreover, its fundamental group is essentially the one of the surface, hence in the case of K3 surfaces the moduli space is an irreducible symplectic variety, while in the case of abelian surfaces, one can obtain an irreducible symplectic variety by taking the fiber of the Albanese map in order to trivialize the fundamental group. If  $\mathcal{M}$  is smooth, as shown in [H], [O1], [Y], it is a deformation of one of the Beauville's examples (see [B1]): the Hilbert scheme of points on a K3 surface or the generalized Kummer variety of an abelian surface. These families give two deformation classes of irreducible symplectic varieties in every even complex dimension. If  $\mathcal{M}$  is singular, then there are two possibilities. Either the singularities result from a bad choice of polarization. Then  $\mathcal{M}$  is birational to one of the above moduli spaces, obtained by changing appropriately the polarization. Or the

singularities cannot be removed by changing the polarization. In this case  $\mathcal{M}$  may not admit a symplectic desingularization, but if it admits one, then it is deformation equivalent to one of the two O'Grady's examples (see [KLS], [KL], [LS]), isolated examples in dimension six (for an abelian surface, see [O3]) and ten (for a K3 surface, see [O2]).

All the known examples of irreducible symplectic varieties are deformations of Lagrangian fibrations. This is a strict analogy with the theory of K3 surfaces, where the elliptic pencils play an important role. Moreover, by a theorem of Matsushita [Ma], a surjective proper morphism with connected fibers from an irreducible symplectic variety onto a variety of smaller dimension, different from a point, is automatically a Lagrangian fibration over a projective space.

In this setting, a natural Lagrangian fibration can be obtained as the relative compactified Jacobian  $\mathcal{J}(|C|)$  of a linear system  $|C|$  on a K3 surface  $S$  (on an abelian surface, as usual, one takes the fiber of the Albanese map inside  $\mathcal{J}(|C|)$ ), see [B2]. It is the moduli space  $\mathcal{M}_S^H(0, C, 1 - g)$  of  $H$ -semistable sheaves on  $S$  of rank zero with  $c_1 = [C]$  and  $\chi = 1 - g$ , and the fibration is given by the support map. When the linear system contains only integral curves, then  $\mathcal{J}(|C|)$  is smooth, and can be deformed to one of Beauville's examples (for an abelian surface the same holds for a fiber of the Albanese map). This is the so called Beauville-Mukai integrable system. In any case, if there are non-integral curves, the singularities only depend on the choice of the polarization. Basing on Sawon's work [S2], it seems plausible that all the Lagrangian fibrations which are relative compactified Jacobians of families of curves, are Beauville-Mukai integrable systems.

Because of this, in order to look for new examples, it is natural to describe Lagrangian fibrations having more general abelian varieties as fibers. The first natural candidates are Prym varieties. This is the idea developed by Markushevitch and Tikhomirov in [MT], where they describe a compactified relative Prymian of dimension four, considering a K3 surface which is the double cover of a Del Pezzo surface of degree two, endowed with the polarization given by the pullback of the anticanonical divisor. The main idea is to interpret such a fibration as a (connected) component of the fixed locus of a symplectic involution of a moduli space  $\mathcal{M}_S^C(0, [C], 1 - g)$ . In this way,  $\mathcal{P}$  inherits a symplectic structure, and the natural map induced by the support map is a Lagrangian fibration. Moreover, using a double cover by  $S^{[2]}$ , they prove that this is the unique symplectic structure and  $\mathcal{P}$  is simply connected. Unfortunately, the choice of a polarization equal to the first Chern class of the sheaves, needed to have a regular involution, gives singularities. From a local analysis via the Kuranishi map, it emerges that these singularities do not admit any symplectic desingularization.

This construction can be adapted to all K3 or abelian surfaces which admit an antisymplectic involution. One can similarly define a symplectic involution on  $\mathcal{M}_S^H(0, C, 1 - g)$ , which is regular if  $H = kC$ . But this choice

implies the existence of singular points. Following this research direction, Arbarello, Saccà and Ferretti in [ASF] have analyzed the case of Enriques surfaces in a general setting, showing that one can obtain new examples of singular Lagrangian fibrations either without any symplectic resolutions, or birational to Beauville-Mukai integrable systems. The problem of studying the other cases remains open. It is known that they are finitely many. In the case of K3 surfaces all the quotients by antisymplectic involutions are either rational surfaces or Enriques surfaces, see [Ma]. In the case of abelian surfaces, they are either bielliptic or ruled surfaces over an elliptic curve.

In this paper, we focus on the natural example to be considered after the one of [MT]: a K3 surface  $S$  which is a double cover of a Del Pezzo of degree three (i.e. a cubic surface), endowed with the polarization  $C$  given by the pullback of the anticanonical divisor.  $\mathcal{M}_S^C(0, [C], -3)$  is a singular irreducible symplectic variety of dimension 8. To construct our example, we consider the  $\tau$ -invariant part of the linear system, denoted by  $|C|^\tau$ .  $\mathcal{P}$  is the compactification in  $\mathcal{M}$  of the natural fibration given by the Prym varieties  $P(C/C')$  for  $C$  smooth, with  $C' := C/\tau$ . These Prym varieties are not principally polarized as in the case of Beauville-Mukai integrable systems, but they have polarization type  $(1, 1, 2)$ . We will prove that  $\mathcal{P}$  is an irreducible symplectic manifold of dimension 6, with only finite quotient singularities, which does not admit any symplectic resolution.

In Section 1, we describe  $S$  and we review, in this example, the general construction of  $\mathcal{P}$  as a component the fixed locus of a symplectic regular involution of  $\mathcal{M}$ . In Section 2, we describe the singularities of  $\mathcal{P}$ , characterizing the reducible members of  $|C|^\tau$  and using the Kuranishi map. We show that it has terminal  $\mathbb{Q}$ -factorial singularities, which implies that it does not admit any symplectic resolution. Indeed a generic singular point  $\mathcal{F}$  is a sheaf supported on two irreducible curves meeting in four points, and a local model of  $\mathcal{P}$  around  $[\mathcal{F}]$  is  $\mathbb{C}^2 \times (\mathbb{C}^4/\pm 1)$ . Moreover, we prove that all the singularities are finite quotient singularities. To do this, we show that the non-generic strictly semistable sheaves  $[\mathcal{F}]$  have as support three irreducible curves, meeting in pairs in two points, and a local model of  $\mathcal{P}$  around  $[\mathcal{F}]$  is  $\mathbb{C}^6/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . In Section 3, we prove that  $\mathcal{P}$  is simply connected and that it has a unique symplectic structure, essentially constructing a rational double cover map from  $S^{[3]}$  to  $\mathcal{P}$ . In Section 4, we compute the degree of the discriminant locus of the Lagrangian fibration, which differs from the classically known ones, and we determine the Euler characteristic of  $\mathcal{P}$ , studying all the singular members of the linear system.

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## 1 Construction of $\mathcal{P}$

Let  $\tau : (x_0, x_1, x_2, x_3, x_4) \mapsto (x_0, x_1, x_2, x_3, -x_4)$  be an involution on  $\mathbb{P}^4$ . Let  $S$  be a K3 surface given by the intersection of a  $\tau$ -invariant quadric  $Y_2$  and a  $\tau$ -invariant cubic  $Y_3$  in  $\mathbb{P}^4$ . Then  $Y_2 : F_2 = x_4^2 + f_2(x_0, x_1, x_2, x_3) = 0$  and  $Y_3 : F_3 = x_4^2 \cdot f_1(x_0, x_1, x_2, x_3) + f_3(x_0, x_1, x_2, x_3) = 0$ . Moreover, since we only care about  $S$ , adding  $-F_2 \cdot f_1$  to the equation of  $Y_3$ , we can suppose that the cubic is a cone with vertex  $p_0 = (0, 0, 0, 0, 1)$ :

$$Y_2 : F_2 = x_4^2 + f_2(x_0, x_1, x_2, x_3) = 0,$$

$$Y_3 : F_3 = f_3(x_0, x_1, x_2, x_3) - f_2(x_0, x_1, x_2, x_3) \cdot f_1(x_0, x_1, x_2, x_3) = 0.$$

Then  $\tau$  induces an involution on  $S$ , which we denote again by  $\tau$ . We can thus consider the double covering map  $\phi : S \rightarrow X := S/\tau$ , ramified along the fixed locus of  $\tau$ , which is the section of  $S$  by the hyperplane  $H_4 : x_4 = 0$ , a curve of genus 4 and degree 6. It is isomorphic to the branch locus. The map  $\phi$  can be realized as restriction of the projection  $\pi : \mathbb{P}^4 \dashrightarrow H_4$  from the point  $p_0$ . Then both  $X$  and  $S$  are surfaces in the same projective space  $\mathbb{P}^4$ , so that  $X = Y_3 \cap H_4$ , and the ramification curve coincides with the branch curve  $B := Y_2 \cap Y_3 \cap H_4$ . Thus  $X$  is a Del Pezzo surface of degree 3. Moreover, since a generic cubic surface is obtained as the blowup of  $\mathbb{P}^2$  in 6 points, for generic  $S$  we have  $\text{Pic}(S) \cong \text{Pic}(X) \cong \mathbb{Z}^7$ , and, taking into account the intersection pairing, we have  $\text{Pic}(S) \cong L_S := I_{1,6}(2)$ .

Let  $|\mathcal{O}_S(1)|^\tau = \langle x_0, x_1, x_2, x_3 \rangle \cong \mathbb{P}^3$  be the linear system of  $\tau$ -invariant hyperplane sections of  $S$ . It is the pullback, via  $\phi$ , of  $|\mathcal{O}_X(1)|$ . Indeed, a curve of  $|\mathcal{O}_S(1)|^\tau$  is a section of  $S$  by a  $\tau$ -invariant hyperplane of  $\mathbb{P}^4$ , which is spanned by a hyperplane of  $H_4$  and by  $p_0$ .

Considering  $X$  as a blowup of  $\mathbb{P}^2$ , we have  $|\mathcal{O}_X(1)| = |3H - E_1 - \dots - E_6|$ , so a curve in this linear system is the strict transform in  $X$  of a plane cubic passing through the 6 base points of the blowup. Moreover the branch locus of  $\phi$  is  $2H_X = 6H - 2E_1 - \dots - 2E_6$ , which is the strict transform of a plane sextic with exactly 6 simple nodes, one in each base point of the blowup.

Choosing  $C \in |\mathcal{O}_S(1)|^\tau$  and the corresponding  $C' \in |\mathcal{O}_X(1)|$ ,  $\phi|_C : C \rightarrow C'$  is a double covering map between a genus 4 curve and a genus 1 curve, ramified in 6 points.

We want to construct an irreducible symplectic variety  $\mathcal{P}$  as a fibration over  $|\mathcal{O}_S(1)|^\tau$ , whose fiber over  $C$  is the Prym variety  $P(C/C')$  of the double cover  $\phi|_C : C \rightarrow C'$ .

Before focusing on our example, we briefly recall the general theory of Prymians (see [Mum]) and of compactified Jacobians (see [A]).

Given a double cover  $\phi|_C : C \rightarrow C'$ , the associated Prym variety  $P(C/C')$  is defined as the connected component of  $\text{Fix}(-\tau^*) \subset J(C)$  containing the zero. When  $C$  is smooth (hence also  $C'$  is),  $J(C)$  is an abelian variety and  $\text{Fix}(-\tau^*)$  has one or two connected components, depending on whether the double cover is ramified or not. When  $C$  is singular,  $J(C)$  is a degeneration of an abelian variety (not necessarily compact) and  $\text{Fix}(-\tau^*)$  can have more connected components. A natural way to compactify  $J(C)$  consists in adding non-invertible polystable sheaves with respect to a fixed polarization  $L$ . The resulting compactified Jacobian  $J_L(C)$  does not depend on the choice of  $L$  when  $C$  is irreducible. In the reducible case, the semistability of a sheaf  $\mathcal{F}$  can be checked only on maximal subsheaves supported on subcurves of  $C$ .

Coming back to our example,  $P(C/C')$  is an abelian 3-fold with a polarization of type  $(1, 1, 2)$ , because there are 6 branch points (see [Mum]). Hence we get a fibration of dimension 6.

We can also define  $\mathcal{P}$  in a global way, following [MT], as a connected component of the fixed locus of a global version  $\eta$  of the involution  $-\tau^*$ , defined on the universal Jacobian  $\mathcal{J}| := \mathcal{J}|_{|\mathcal{O}_S(1)|^\tau}$  over  $|\mathcal{O}_S(1)|^\tau$  (indeed it is defined on all  $|\mathcal{O}_S(1)|$ ). One advantage of this global point of view is that  $\eta$  is a symplectic involution, so  $\mathcal{P}$  inherits its symplectic structure from that of  $\mathcal{J}$ . In order to define  $\eta$ , we identify  $\mathcal{J}$  with  $\mathcal{M}_S^C(v)$ , where  $\mathcal{M}_S^C(v)$  is the moduli space of  $C$ -semistable sheaves  $\mathcal{F}$  on  $S$  with Mukai vector

$$v = (rk(\mathcal{F}), c_1(\mathcal{F}), c_1(\mathcal{F})^2/2 - c_2(\mathcal{F}) + rk(\mathcal{F})) = (0, C, -3) \in H^{2*}(S).$$

We remark that this choice of the polarization on  $S$  is not  $v$ -generic in the sense of Yoshioka (see [Y]), so we have a singular moduli space. On  $\mathcal{M}_S^C(v)$ , the natural involution  $\tau^*$  generalizes the corresponding involution on  $J(C)$ , and it is antisymplectic. For an invertible sheaf  $\mathcal{F}$  on  $C$ , the global version of the involution  $-1 : \mathcal{F} \mapsto \mathcal{H}om_C(\mathcal{F}, \mathcal{O}_C)$  on  $J(C)$ , can be given by  $j(\mathcal{F}) = \mathcal{H}om_S(\mathcal{F}, \mathcal{O}_C)$ . But  $\chi(\mathcal{F})$  and  $\chi(\mathcal{H}om_S(\mathcal{F}, \mathcal{O}_C))$  may differ when  $C$  is singular and  $\mathcal{F}$  is non-invertible, hence the above formula does not provide a well defined involution on  $\mathcal{M}_S^C(v)$ . To correct this definition, we use the identification

$$\mathcal{H}om_S(\mathcal{F}, \mathcal{O}_C) = \mathcal{E}xt_S^1(\mathcal{F}, \mathcal{O}_S(-C)),$$

valid for an invertible sheaf  $\mathcal{F}$  on  $C$ . It can be obtained by using local isomorphisms  $\mathcal{F}|_U \cong \mathcal{O}|_U$  and by gluing together the isomorphisms

$$\mathcal{H}om_S(\mathcal{O}_C|_U, \mathcal{O}_C|_U) = \mathcal{E}xt_S^1(\mathcal{O}_C|_U, \mathcal{O}_S(-C|_U)),$$

coming from the short exact sequence

$$0 \rightarrow \mathcal{O}_S(-C) \rightarrow \mathcal{O}_S \rightarrow \mathcal{O}_C \rightarrow 0$$

via the application of the functor  $\mathcal{H}om_S(\_, \mathcal{O}_S(-C))$ .

The functor  $\mathcal{E}xt_S^1$  commutes with base change in families (see [AK]), so the global version of  $-1$  can be defined on  $\mathcal{M}_S^C(v)$  by the formula

$$j(\mathcal{F}) := \mathcal{E}xt_S^1(\mathcal{F}, \mathcal{O}_S(-C)).$$

$j$  respects the stability when the support of a sheaf is integral, because such a sheaf is stable with respect to any polarization. So it is defined as a rational involution also for a  $v$ -generic choice of the polarization. For our choice of the polarization, it is also a regular involution, because if  $0 \rightarrow \mathcal{G} \rightarrow j(\mathcal{F})$ , then we get, applying  $j$ ,  $\mathcal{F} \rightarrow j(\mathcal{G}) \rightarrow 0$ , and  $\mu_C(\mathcal{G}) \leq \mu_C(j(\mathcal{F}))$  is equivalent to  $\mu_C(\mathcal{F}) \leq \mu_C(j(\mathcal{G}))$ . Finally,  $j$  is antisymplectic, as shown in [ASF], Proposition 3.11.

Then  $\eta := j \circ \tau^*$  is an involution on  $\mathcal{M}_S^C(v)$ , because  $j$  and  $\tau^*$  commute and  $\chi(\eta(\mathcal{L})) = \chi(\mathcal{L})$ ,  $c_1(\eta(\mathcal{L})) = c_1(\mathcal{L})$ . Moreover it is symplectic.  $\mathcal{P}$  is defined as the connected component of  $Fix(\eta) \subset \mathcal{M}_S^C(v)$  containing the zero section.

So  $\mathcal{P}$  inherits a symplectic form from  $\mathcal{M}_S^C(v)$ , and the natural map  $\mathcal{P} \rightarrow |\mathcal{O}_S(1)|^\tau$  is a Lagrangian fibration. We will see that the symplectic form is unique and that  $\mathcal{P}$  is simply connected, so that  $\mathcal{P}$  is a singular generalization of symplectic variety.

## 2 Singularities of $\mathcal{P}$

According to Mukai (see [Mu]), the stable locus of  $\mathcal{M}_S^C(v)$  is smooth, so that the singular locus is contained in the locus of strictly semistable sheaves.

Moreover, in our case, a semistable sheaf in  $\mathcal{J}$  supported on an integral curve is always stable, because any rank 1 torsion free sheaf on an integral curve has no proper subsheaves and hence has no destabilizing subsheaves. So every strictly semistable sheaf  $\mathcal{F}$  is supported on a non-integral curve in  $|\mathcal{O}_S(1)|^\tau$ .

Hence, the singularities of  $\mathcal{P}$  are contained in the locus of  $\eta$ -invariant  $C$ -strictly semistable sheaves in  $\mathcal{J}_1$ , which are necessarily supported on a reducible curve.

**Lemma 2.1.**  *$C \in |\mathcal{O}_S(1)|^\tau$  is reducible if and only if  $C' \in |\mathcal{O}_X(1)|$  is reducible. Moreover, all the elements of the two linear systems are reduced.*

*Proof.* Clearly, if  $C'$  is reducible, then also  $C$  is.

Suppose  $C'$  irreducible and  $C$  reducible. Then  $C'$  is obtained by intersecting  $X$  with a hyperplane  $H \in (\mathbb{P}^3)^*$ , totally tangent to  $B$ . In fact if  $\phi|_C : C \rightarrow C'$  has a simple ramification point, corresponding to a point of transversal intersection of  $H$  and  $B$ , then  $C$  is irreducible for topological reasons. Since the arithmetic genus of  $C$  is 4,  $C$  is the union of two smooth curves of genus 1 meeting in 3 points. But  $C = S \cap \langle H, P_0 \rangle \subset Y_2 \cap \langle H, P_0 \rangle$ , the latter intersection being a quadric in  $\mathbb{P}^3$ . A smooth genus 1 curve is in the linear

system of  $\mathcal{O}(2)$  on a quadric, while  $C$  belongs to  $|\mathcal{O}(3)|$ , which is absurd. Finally,  $C$  is non-reduced if and only if  $C'$  is so, but on a generic cubic surface ( $F_3$  is a generic polynomial in  $x_0, x_1, x_2, x_3$ ) there are no non-reduced curves.  $\square$

Hence we obtain the following characterization.

**Proposition 2.2.** *Using the natural embedding  $X \rightarrow |\mathcal{O}_X(1)|^*$ , the reducible members of  $|\mathcal{O}_S(1)|^\tau$  are parametrized by the 27 lines dual to the 27 lines on  $X$ . They meet in exactly 16 triple points, and each line meets exactly other 10 lines. A point lying on only one line represents  $C = C_1 \cup C_2$ , where  $C_1$  and  $C_2$  are smooth curves of genus respectively 0 and 1 intersecting in 4 points; moreover  $C' = C'_1 \cup C'_2$ , where  $C'_1$  and  $C'_2$  are a line and a conic lying on a plane, hence meeting in 2 points. A point in the intersection of 3 dual lines represents  $C = C^1 \cup C^2 \cup C^3$ , with  $C^1, C^2, C^3$  smooth rational curves, intersecting each other in 2 points; moreover,  $C' = C^{1'} \cup C^{2'} \cup C^{3'}$  is a triangle formed by 3 lines lying on a plane. The triple intersection points represent the curves  $C$  with 3 irreducible components that can be obtained as degenerations of curves with 2 irreducible components by fixing one of them and deforming the other 2.*

*Proof.* By the previous lemma, it is enough to determine the reducible members  $C'$  of  $|\mathcal{O}_X(1)|$ . We describe geometrically these curves. As  $Y_3$  is a cubic cone, it contains 27 planes, each one generated by  $P_0$  and one of the 27 lines on  $Y_3 \cap H_4$ . If one of these lines is given by  $H' \cap X$ ,  $H = \langle H', P_0 \rangle \in (\mathbb{P}^4)^*$  is a  $\tau$ -invariant hyperplane containing one of these planes. Then  $H \cap Y_3$  is the union of the plane itself with either a quadric surface or other two planes. Restricting to  $Y_2 \cap H$ , which is smooth for a generic choice of  $f_2$  and hence isomorphic to  $\mathbb{P}^1 \times \mathbb{P}^1$ , we obtain a reducible  $C \in |\mathcal{O}(1)|^\tau$ , union either of a  $(1, 1)$  curve  $C_1$  and a  $(2, 2)$  curve  $C_2$  on  $\mathbb{P}^1 \times \mathbb{P}^1$ , or of  $(1, 1)$  curves  $C^1, C^2, C^3$ . Moreover  $C_1 \cdot C_2 = 4$ , while  $C^i \cdot C^j = 2$ . Since  $C$  has 6 ramification points, the only possibilities are  $C' = C'_1 \cup C'_2$ , with  $C'_1$  and  $C'_2$  two genus zero curves meeting in 2 points, or  $C' = C^{1'} \cup C^{2'} \cup C^{3'}$ , with  $C^{i'}$  genus zero curve.

In fact, if  $C'$  is reducible, then it is the union either of a conic and a line in  $X$ , or of three lines in  $X$ .

Hence for each line in  $X$ , we get a line of reducible elements in  $|\mathcal{O}_S(1)|^\tau$ . Indeed, in  $\mathbb{P}^3$  there is a pencil of planes containing a fixed line.

From the configuration of the 27 lines in a cubic surface, we can deduce the configuration of these dual lines, which are the orthogonals to the lines in  $X$ . Their intersection points correspond obviously to the degeneration when  $C'$  becomes the union of 3 lines. Considering  $X$  as the blowup in 6 points  $p_1, \dots, p_6$  of  $\mathbb{P}^2$ , we call  $E_i$  the exceptional divisor of  $p_i$ ,  $F_{ij}$  the pullback of the line  $\langle p_i, p_j \rangle$  and  $G_i$  the pullback of a conic passing through  $P_j$  for all  $i \neq j$ . Then we get  $C' = F_{ij} \cup F_{kl} \cup F_{mn}$  with  $i, j, k, l, m, n$  all distinct, or  $C' = F_{ij} \cup E_i \cup G_j$ . Thus, each line meets other 10 lines in different points,



and there are in total 48 elements of the linear system with 3 irreducible components. Since the dual of 3 coplanar lines in  $\mathbb{P}^3$  is given by 3 lines in  $\mathbb{P}^3$  meeting in one point, there are 16 points where 3 of the 27 orthogonal lines intersect, and no other intersection points. When  $C'$  is given by 3 lines, it can deform to a conic and a line in 3 different ways, by fixing one line and deforming the other 2 to a smooth conic by rotating a plane through the fixed line. The triple points of the dual configuration of lines are either the intersection of  $F_{ij}^*, F_{kl}^*, F_{mn}^*$  with  $i, j, k, l, m, n$  all distinct, or the intersection of  $F_{ij}^*, E_i^*, G_j^*$ , and there are no other intersections. Hence each dual line meets other 10 dual lines in 5 points, and each intersection point is triple.  $\square$

*Remark 2.3.* A geometric description of the reducible curves  $C'$  can also be obtained by viewing  $X$  as the blowup of  $\mathbb{P}^2$  in  $p_1, \dots, p_6$ . A generic reducible element of  $|3H - p_1 - \dots - p_6|_{\mathbb{P}^2}$  is the union of a conic and a line passing through the 6 base points of the blowup. They form 21 pencils in  $|\mathcal{O}_X(1)| = |3H - E_1 - \dots - E_6|$ : 6  $\mathbb{P}^1$ 's, denoted by  $L_i$  with  $i = 1, \dots, 6$ , are given by a line passing through  $p_i$  and the conic passing through the remaining 5 base points; 15  $\mathbb{P}^1$ 's, denoted by  $L_{jk}$  with  $1 \leq j < k \leq 6$ , are given by the line spanned by  $p_j$  and  $p_k$ , and a conic through the remaining 4 points. The remaining 6  $\mathbb{P}^1$ 's, parametrizing reducible curves of  $|\mathcal{O}_X(1)|$ , correspond to the 6 pencils, denoted by  $M_i$ , parametrizing the strict transforms of cubics, singular at  $p_i$ , plus the exceptional divisor  $E_i$ .

The curves  $C'$  with 3 irreducible components are described as follows. They correspond to the strict transforms on  $X$  of the unions of three lines passing through the 6 base points of the blowup, and of the unions of a conic through 5 base points with a line through one of the same 5 base points and through the sixth one. These curves correspond to the intersections of the 27 lines. More precisely,  $L_i \cdot L_j = 0$ ;  $L_i \cdot L_{jk}$  is 1 if  $i \in \{j, k\}$  and 0 otherwise;  $L_i \cdot M_j$  is 1 if  $i \neq j$  and 0 otherwise;  $L_{jk} \cdot L_{hi}$  is 1 if  $\#\{h, i, j, k\} = 4$  and 0 otherwise;  $L_{jk} \cdot M_i$  is 1 if  $i \in \{j, k\}$  and 0 otherwise;  $M_i \cdot M_j = 0$ . In particular, we obtain again that all the intersection points of the 27 lines are triple points.

Now we can determine the singular locus of  $\mathcal{J}_1 := \mathcal{J}|_{|\mathcal{O}_S(1)|^\tau}$  and of  $\mathcal{P}$ .

**Lemma 2.4.** *All the strictly semistable sheaves in  $\mathcal{J}_1$  are supported on the reducible curves of  $|\mathcal{O}_S(1)|^\tau$ . For each curve  $C = C_1 \cup C_2$ , there is a 1-dimensional family of  $S$ -equivalence classes of strictly semistable sheaves, given by  $[\mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2]$ , with  $\mathcal{F}_2 \in J^{-2}(C_2)$ . For each curve  $C = C^1 \cup C^2 \cup C^3$ , there are three 1-dimensional families of  $S$ -equivalence classes of strictly semistable sheaves, given by  $[\mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2]$ , with  $\mathcal{F}_2 \in J^{(-1, -1)}(C_2) = \mathbb{C}^*$ ,  $C_1$  an irreducible component of  $C$  and  $C_2$  the union of the remaining two irreducible components; these families meet quasi-transversely in one point, represented by  $[\mathcal{O}_{C^1}(-2) \oplus \mathcal{O}_{C^2}(-2) \oplus \mathcal{O}_{C^3}(-2)]$ .*



*Proof.* As explained before, if  $\mathcal{F} \in \mathcal{J}_1$  is supported on an integral curve  $C$ , then any rank 1 torsion free sheaf on  $C$  is stable, both as a sheaf on  $C$  and on  $S$ . So a strictly semistable sheaf  $\mathcal{F}$  is supported on a non-integral curve  $C \in |\mathcal{O}_S(1)|^\tau$ , i.e. on a reduced reducible curve by the previous lemma. If  $C = C_1 \cup C_2$ , where  $C_i$  has genus  $g_i$ , then  $\mathcal{F} \in \text{Ext}_S^1(\mathcal{F}_1, \mathcal{F}_2) = \mathbb{C}^{C_1 \cdot C_2} = \mathbb{C}^4$ , where  $\mathcal{F}_i$  is a pure 1-dimensional sheaf on  $C_i$  (which is stable since it is invertible on  $C_i$ ) of degree  $d_i$ . Thus

$$\mu_C(\mathcal{F}_1) = \mu_C(\mathcal{F}_2),$$

and since

$$\mu_C(\mathcal{F}_i) = \frac{\chi(\mathcal{F}_i)}{C_i \cdot C} = \frac{1 - g_i + d_i}{C_i^2 + C_1 \cdot C_2} = \frac{1 - g_i + d_i}{2g_i - 2 + C_1 \cdot C_2},$$

we get

$$\frac{1 + d_1}{2} = \frac{d_2}{4}, \quad \text{i.e.} \quad 2d_1 + 2 = d_2.$$

Moreover  $\chi(\mathcal{F}(n)) = \chi(\mathcal{F}_1(n)) + \chi(\mathcal{F}_2(n))$ , i.e.  $-4 = d_1 + d_2$ . Hence  $d_1 = d_2 = -2$ , and  $\mathcal{F}_1 = \mathcal{O}_{C_1}(-2)$ ,  $\mathcal{F}_2 \in J^{-2}(C_2)$ . So  $0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}$  gives a Jordan-Hölder filtration of  $\mathcal{F}$  and  $[\mathcal{F}] = [\mathcal{F}_1 \oplus (\mathcal{F}/\mathcal{F}_1)] = [\mathcal{F}_1 \oplus \mathcal{F}_2]$  by the exactness of  $0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F} \rightarrow \mathcal{F}_1 \rightarrow 0$ .

If  $C = C^1 \cup C^2 \cup C^3$ , then it comes from a degeneration of a  $C_2$  in 2 rational curves, which we assume to be  $C^1$  and  $C^2$  (hence in this notation  $C^3$  is identified with  $C_1$ ). Then we still have the exact sequence  $0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F} \rightarrow \mathcal{O}_{C_1}(-2) \rightarrow 0$ , with  $\mathcal{F}_2 \in J^{-2}(C_2)$  semistable but not necessarily stable (because it is a limit of stable sheaves on a family of  $C_2$ 's degenerating to two rational curves). Again  $\mathcal{F}_2 \in \text{Ext}_S^1(\mathcal{G}_1, \mathcal{G}_2) = \mathbb{C}^{C^1 \cdot C^2} = \mathbb{C}^2$ , where  $\mathcal{G}_i$  is a pure 1-dimensional sheaf on  $C^i$  of degree  $d^i$ . So

$$\mu_C(\mathcal{G}_2) \leq \mu_C(\mathcal{F}_2) \leq \mu_C(\mathcal{G}_1),$$

and since

$$\mu_C(\mathcal{G}_i) = \frac{\chi(\mathcal{G}_i)}{C^i \cdot C} = \frac{1 + d^i}{(C^i)^2 + C^i \cdot (C - C^i)} = \frac{1 + d^i}{2},$$

we get

$$\frac{1 + d^2}{2} \leq \frac{-2}{4} \leq \frac{1 + d^1}{2}, \quad \text{i.e.} \quad d^2 \leq -2 \leq d^1.$$

Moreover  $\chi(\mathcal{F}_2(n)) = \chi(\mathcal{G}_1(n)) + \chi(\mathcal{G}_2(n))$ , i.e.  $d^1 + d^2 = -4$ .

Thus  $(d^1, d^2)$  can be  $(-1, -3)$  (and  $\mathcal{F}_2$  is stable),  $(-2, -2)$  (and  $\mathcal{F}_2$  is strictly semistable) or  $(0, -4)$  ( $\mathcal{F}_2$  is again strictly semistable with a maximal destabilizing subsheaf  $\mathcal{G}_1(-C_1 \cdot C_2)$ , so that  $\mathcal{F}_2$  is considered as an extension given by an element of  $\text{Ext}_S^1(\mathcal{G}_2, \mathcal{G}_1)$ ).

So either  $\mathcal{F}$  admits a Jordan-Hölder filtration  $0 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}$  and  $[\mathcal{F}] = [\mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2]$  with  $\mathcal{F}_2 \in J^{(-1,-1)}(C_2) = \mathbb{C}^*$ ; or  $\mathcal{F}$  admits a Jordan-Hölder filtration  $0 \rightarrow \mathcal{G}_2 \rightarrow \mathcal{F}_2 \rightarrow \mathcal{F}$ , and  $[\mathcal{F}] = [\mathcal{O}_{C^1}(-2) \oplus \mathcal{O}_{C^2}(-2) \oplus \mathcal{O}_{C^3}(-2)]$ .

The geometric meaning of this degeneration is the following: identifying  $\mathcal{F}_2 \in J^{(-1,-1)}(C_2)$  as the gluing  $\mathcal{O}_{C^1}(-p_1) \# \mathcal{O}_{C^2}(-p_2)$ , with  $p_i$  point on  $C^i$ , when  $p_1$  (respectively  $p_2$ ) tends to one of the two singular points, we obtain as limit  $\mathcal{O}_{C^1} \# \mathcal{O}_{C^2}(-p_1 - p_2) \in J^{(0,-2)}(C_2)$  (respectively  $\mathcal{O}_{C^1}(-p_1 - p_2) \# \mathcal{O}_{C^2} \in J^{(-2,0)}(C_2)$ ).  $\square$

**Lemma 2.5.** *All the strictly semistable sheaves in  $\mathcal{J}_1$  are  $\eta$ -invariant.*

*Proof.* Using the previous lemma, it is enough to determine the strictly semistable sheaves  $\mathcal{F} \in \mathcal{J}_1$  which are  $\eta$ -invariant. Considering a sheaf  $\mathcal{F}$  with support  $C = C_1 \cup C_2$ , we have

$$\begin{aligned} \eta(\mathcal{F}) &= j(\tau^*(\mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2)) = \\ &= j(\tau^*(\mathcal{O}_{C_1}(-2)) \oplus \tau^*(\mathcal{F}_2)) = \\ &= j(\mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2^*(-p_1 - \dots - p_4)) = \\ &= \mathcal{E}xt_S^1(\mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2^*(-p_1 - \dots - p_4), \mathcal{O}_S(-C)) = \\ &= \mathcal{E}xt_S^1(\mathcal{O}_{C_1}(-2), \mathcal{O}_S(-C)) \oplus \mathcal{E}xt_S^1(\mathcal{F}_2^*(-p_1 - \dots - p_4), \mathcal{O}_S(-C)) = \\ &= (\mathcal{O}_{C_1}(2) \otimes \mathcal{O}_S(-C) \otimes \omega_{C_1}) \oplus (\mathcal{F}_2(p_1 + \dots + p_4) \otimes \mathcal{O}_S(-C) \otimes \omega_{C_2}) = \\ &= \mathcal{O}_{C_1}(-2) \oplus \mathcal{F}_2 = \mathcal{F}, \end{aligned}$$

where the second equality comes from the fact that  $\tau$  acts separately on each irreducible component; in the third one,  $p_1, \dots, p_4$  are the singular points of  $C$ ; in the sixth one, we use the identities for a sheaf  $\mathcal{F}$  with support  $C$

$$\mathcal{E}xt_S^1(\mathcal{F}, \mathcal{G}) = \mathcal{F}^* \otimes \mathcal{G} \otimes \mathcal{N}_{C/S}, \quad \mathcal{N}_{C/S} = \mathcal{O}_S(C)|_C = \omega_C;$$

in the seventh one we use the identification

$$\mathcal{O}_S(-C)|_{C_i} = \mathcal{O}_{C_i}(-C_1 \cdot C_2) \oplus \mathcal{O}_{C_i}(-C_i^2).$$

Moreover, when  $\mathcal{F}$  has support  $C = C^1 \cup C^2 \cup C^3$ , then by deformation considerations we also have  $\eta(\mathcal{F}) = \mathcal{F}$ .

So every strictly semistable sheaf in  $\mathcal{J}_1$  is  $\eta$ -invariant.  $\square$

**Corollary 2.6.**  *$\mathcal{P}$  has symplectic singularities, i.e. its symplectic form can be extended to a regular form on a resolution of singularities.*

*Proof.* Since the codimension of the singular locus is 4, this follows from Flenner's theorem (see [F]).  $\square$

Hence, proving that  $\mathcal{P}$  is simply connected and  $h^{(2,0)}(\mathcal{P}) = 1$ , we have, from the previous result, that  $\mathcal{P}$  admits a Beauville-Bogomolov form.

To describe the singularities of  $\mathcal{P}$ , we describe, locally around the origin, the tangent cone  $C_{[\mathcal{F}]}(\mathcal{P})$  to  $\mathcal{P}$  at the strictly semistable sheaves  $\mathcal{F}$ . To this aim, we use the Kuranishi map to determine a local analytic model of  $\mathcal{J}$  at the strictly semistable sheaves, given by its tangent cone  $C_{[\mathcal{F}]}(\mathcal{J})$ , see [LS].

The Kuranishi map  $k : Ext_S^1(\mathcal{F}, \mathcal{F}) \rightarrow Ext_S^2(\mathcal{F}, \mathcal{F})$  is a formal map satisfying the following properties:

- 1) its image is contained in the kernel of the trace map

$$tr : Ext_S^2(\mathcal{F}, \mathcal{F}) \rightarrow H^2(\mathcal{O}_S) = \mathbb{C},$$

denoted by  $Ext_S^2(\mathcal{F}, \mathcal{F})_0$ ;

- 2)  $k$  is equivariant with respect to the natural conjugation action of  $G := PAut(\mathcal{F})$  on  $Ext_S^1(\mathcal{F}, \mathcal{F})$  and  $Ext_S^2(\mathcal{F}, \mathcal{F})$ ;

- 3)  $(k^{-1}(0)//G, 0)$  is a local analytic model of  $(\mathcal{M}_S^C(v), \mathcal{F})$ ;

- 4)  $k$  has an expansion at 0

$$k = k_2 + k_3 + \dots$$

starting from a quadratic term which is the cup product

$$k_2(\mathcal{G}) := \frac{1}{2}\mathcal{G} \cup \mathcal{G}.$$

Using this map one can easily show that stable sheaves over a symplectic surface give smooth points of the corresponding moduli space. We briefly review this basic result. Since a K3 has trivial canonical bundle, the trace map  $tr : Ext_S^2(\mathcal{F}, \mathcal{F}) \rightarrow H^2(\mathcal{O}_S) = \mathbb{C}$  is the Serre dual of the natural map  $H^0(\mathcal{O}_S) = \mathbb{C} \rightarrow Hom_S(\mathcal{F}, \mathcal{F})$ . If  $\mathcal{F}$  is stable, this map is an isomorphism, hence  $Ext_S^2(\mathcal{F}, \mathcal{F})_0 = 0$  and  $G = \mathbb{C}^*$  acts trivially on  $Ext_S^1(\mathcal{F}, \mathcal{F})$ , so, by 3),  $Ext_S^1(\mathcal{F}, \mathcal{F})$  is a local model of  $\mathcal{M}_S^C(v)$  at  $\mathcal{F}$ . Since a tangent vector of a moduli space of sheaves at a stable point  $\mathcal{F}$  corresponds to a family of sheaves over  $Spec(\mathbb{C}[t]/(t^2))$  with  $\mathcal{F}$  as central fiber, we can identify the tangent space to such a moduli space with  $Ext_S^1(\mathcal{F}, \mathcal{F})$ . Combining these considerations, we get that the stable points are smooth.

Let us focus now on strictly semistable sheaves. Combining 3) and 4), we see that the tangent cone  $C_{[\mathcal{F}]}(\mathcal{J})$  at  $[\mathcal{F}]$  is locally analytically equivalent to  $(k_2^{-1}(0)//G, 0)$ . We have

$$C_{[\mathcal{F}]}(\mathcal{P}) = C_{[\mathcal{F}]}(\mathcal{J}^\eta) \subset C_{[\mathcal{F}]}(\mathcal{J})^{\eta^*} \subset (Ext_S^1(\mathcal{F}, \mathcal{F})//G)^{\eta^*},$$

and in the following two propositions we show that these inclusions are equalities in the case of the strictly semistable sheaves we are considering, and we explicitly describe the type of singularities.

**Proposition 2.7.** *At a polystable sheaf  $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ ,  $C_{[\mathcal{F}]}(\mathcal{P})$  is locally analytically equivalent to  $\mathbb{C}^2 \times (\mathbb{C}^4 / \pm 1)$ .*

*Proof.* 1) We start describing  $Ext_S^1(\mathcal{F}, \mathcal{F})$ .

Let us set

$$\begin{aligned} U_1 &:= Ext_S^1(\mathcal{F}_1, \mathcal{F}_1), \quad U_2 := Ext_S^1(\mathcal{F}_2, \mathcal{F}_2), \\ W &:= Ext_S^1(\mathcal{F}_1, \mathcal{F}_2), \quad W^* := Ext_S^1(\mathcal{F}_2, \mathcal{F}_1). \end{aligned}$$

Since  $C_1$  is a rational curve, it is rigid and the sheaves on it form a discrete set, so  $\mathcal{F}_1$  can't be deformed and  $U_1 = H^0(\mathcal{N}_{C_1/S}) \oplus Ext_{C_1}^1(\mathcal{F}_1, \mathcal{F}_1) = 0$ . Since  $C_2$  has genus 1,  $\mathcal{F}_2$  can be deformed either as a sheaf in  $J^{-2}(C_2)$  or by varying its support, the two possibilities corresponding to the two summands in  $U_2 = H^0(\mathcal{N}_{C_2/S}) \oplus Ext_{C_2}^1(\mathcal{F}_2, \mathcal{F}_2) = \mathbb{C}^2$ . Finally, as the supports of  $\mathcal{F}_1$  and  $\mathcal{F}_2$  are transversal, we get  $W = \mathbb{C}^{C_1 \cdot C_2} = \mathbb{C}^4$ . So

$$Ext_S^1(\mathcal{F}, \mathcal{F}) = U_1 \oplus U_2 \oplus W \oplus W^* = \mathbb{C}^{10}.$$

Choosing coordinates  $x_1, \dots, x_4$  in  $W$  such that  $\tau$  exchanges  $x_1 \leftrightarrow x_2$  and  $x_3 \leftrightarrow x_4$ , let  $y_1, \dots, y_4$  be the dual coordinates in  $W^*$ . Let  $z_1, z_2$  be coordinates in  $U_2$ .

$Aut(\mathcal{F}) = Aut(\mathcal{F}_1) \oplus Aut(\mathcal{F}_2) = \mathbb{C}^{*2}$  by the stability of  $\mathcal{F}_i$ , hence  $G = \mathbb{C}^*$ , and, since  $(\lambda_1, \lambda_2) \cdot (\underline{x}, \underline{y}) = (\lambda_1 \lambda_2^{-1} \underline{x}, \lambda_1^{-1} \lambda_2 \underline{y})$  for  $(\lambda_1, \lambda_2) \in Aut(\mathcal{F})$ , its action on  $Ext_S^1(\mathcal{F}, \mathcal{F})$  is  $\lambda \cdot (z, \underline{x}, \underline{y}) = (z, \lambda \underline{x}, \lambda^{-1} \underline{y})$ , where  $\lambda = \lambda_1 / \lambda_2$ .

The algebra of invariants of the action of  $G$  on  $\mathbb{P}(Ext_S^1(\mathcal{F}, \mathcal{F}))$  is generated by the quadratic monomials  $u_{ij} = x_i y_j$ , plus two coordinates  $z_1, z_2$  on which  $G$  acts trivially, and the generating relations are the quadratic ones

$$u_{ij} u_{kl} = u_{kj} u_{il}.$$

Hence we get the Segre embedding of  $\mathbb{P}^3 \times \mathbb{P}^3$  into  $\mathbb{P}^{15}$ . In conclusion,  $Ext_S^1(\mathcal{F}, \mathcal{F}) / G = \mathbb{C}^2 \times Z$ , where  $Z$  is the affine cone over the Segre embedding.

2) Now we will describe the fixed locus of  $\eta^*$  in  $C_{[\mathcal{F}]}(\mathcal{J})$ .

On  $U_2$ ,  $j$  acts as  $\tau$  (the  $-1$  involution on an elliptic curve). On  $W \oplus W^*$ , with the above choice of the coordinates  $\underline{x}$  and  $\underline{y}$ ,  $j$  exchanges  $x_i \leftrightarrow y_i$ . Hence the action of  $\eta^*$  is

$$\eta^*(z, \underline{x}, \underline{y}) = (z, y_2, y_1, y_4, y_3, x_2, x_1, x_4, x_3).$$

Its fixed locus is then

$$x_1 = y_2, x_2 = y_1, x_3 = y_4, x_4 = y_3.$$

Since

$$\eta^*(\lambda \cdot (z, \underline{x}, \underline{y})) = \frac{1}{\lambda} \eta^*(z, \underline{x}, \underline{y}),$$

$\eta^*$  induces a well defined involution on  $\text{Ext}_S^1(\mathcal{F}, \mathcal{F})/G$ . From the above formula we also see that  $\eta^*$  is not  $G$ -invariant, hence its fixed locus on the quotient cannot be described as quotient of its fixed locus. To characterize it, we observe that in the invariant coordinates  $u_{ij} = x_i y_j$ , these conditions become

$$u_{11} = u_{22}, u_{33} = u_{44}, u_{13} = u_{42}, u_{14} = u_{32}, u_{23} = u_{41}, u_{24} = u_{31}.$$

The quadratic relations in  $u_{ij}$ , combined with these ones, give the equations of the Veronese image of  $\mathbb{P}^3$  in  $\mathbb{P}^9$ . Thus  $(\text{Ext}_S^1(\mathcal{F}, \mathcal{F})/G)^{\eta^*} = \mathbb{C}^2 \times W$ , where  $W$  is the affine cone over the Veronese image of  $\mathbb{P}^3$ , and by dimension reasons we deduce  $C_{[\mathcal{F}]}(\mathcal{P}) = \mathbb{C}^2 \times W$ . Alternatively, one can describe  $W$  as the quotient  $\mathbb{C}^4 / \pm 1$ . By choosing coordinates  $w_1, \dots, w_4$  on  $\mathbb{C}^4$  with the action of  $-1$  given by  $w_i \mapsto -w_i$ , the algebra of invariant functions has 10 generators

$$v_{ij} := w_i w_j,$$

and 72 relations

$$v_{ij} v_{kl} = v_{kj} v_{il},$$

which describe exactly  $W$ . □

**Corollary 2.8.**  *$\mathcal{P}$  does not admit any symplectic desingularization.*

*Proof.* By the above proposition,  $\mathcal{P}$  is locally analytically isomorphic to  $\mathbb{C}^2 \times (\mathbb{C}^4 / -1)$  around a polystable sheaf  $\mathcal{F} = \mathcal{F}_1 \oplus \mathcal{F}_2$ . The singularity  $\mathbb{C}^4 / -1$  is known to be  $\mathbb{Q}$ -factorial, so it has no small resolutions, and terminal (see [MS]), that is the canonical sheaf of any resolution of singularities contains all the exceptional divisors with strictly positive coefficients. Thus none of the resolutions has trivial canonical class, and hence none of them is symplectic. □

*Remark 2.9.* It is interesting to note that  $\mathcal{J}$  admits a symplectic resolution while  $\mathcal{P}$  does not.

**Corollary 2.10.**  *$\mathcal{P}$  cannot be deformed to a smooth symplectic 6-fold. In particular, it is not deformation equivalent to the Hilbert scheme of 3 points on a K3 surface, neither to a generalized Kummer 6-fold, nor to the O'Grady example of dimension 6.*

*Remark 2.11.* It can be interesting to determine its Beauville-Bogomolov form. An example of a BB-form of a singular irreducible symplectic variety is described by Menet in [Me]: he computes it for the singular 4-fold of [MT].

*Remark 2.12.* The symplectic involution on  $\mathcal{J}$  extends only to a rational involution on a resolution of singularities  $\tilde{\mathcal{J}}$ . Indeed, a regular involution on  $\tilde{\mathcal{J}}$  would have, as its fixed locus  $\tilde{\mathcal{P}}$ , a resolution of singularities of  $\mathcal{P}$ , and it would induce a symplectic form on it, which is absurd by the previous result. This gives a rational symplectic involution on an irreducible symplectic manifold deformation equivalent to  $S^{[3]}$ .

**Proposition 2.13.** *At a polystable sheaf  $\mathcal{F} = \mathcal{O}_{C^1}(-2) \oplus \mathcal{O}_{C^2}(-2) \oplus \mathcal{O}_{C^3}(-2)$ ,  $C_{[\mathcal{F}]}(\mathcal{P})$  is locally analytically equivalent to  $\mathbb{C}^6/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ , where the action on  $\mathbb{C}^6$  is given by  $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle (1, 1, -1, -1, -1, -1), (-1, -1, 1, 1, -1, -1) \rangle$ .*

*Proof.* 1) We set

$$\begin{aligned} W_{12} &:= \text{Ext}_S^1(\mathcal{O}_{C^1}(-2), \mathcal{O}_{C^2}(-2)), & W_{12}^* &:= \text{Ext}_S^1(\mathcal{O}_{C^2}(-2), \mathcal{O}_{C^1}(-2)), \\ W_{13} &:= \text{Ext}_S^1(\mathcal{O}_{C^1}(-2), \mathcal{O}_{C^3}(-2)), & W_{13}^* &:= \text{Ext}_S^1(\mathcal{O}_{C^3}(-2), \mathcal{O}_{C^1}(-2)), \\ W_{23} &:= \text{Ext}_S^1(\mathcal{O}_{C^2}(-2), \mathcal{O}_{C^3}(-2)), & W_{23}^* &:= \text{Ext}_S^1(\mathcal{O}_{C^3}(-2), \mathcal{O}_{C^2}(-2)). \end{aligned}$$

As before, since the supports of  $\mathcal{O}_{C^i}(-2)$  and  $\mathcal{O}_{C^j}(-2)$  are transversal for  $i \neq j$ , we get  $W_{ij} = \mathbb{C}^{C^1 \cdot C^2} = \mathbb{C}^2$ . So

$$\text{Ext}_S^1(\mathcal{F}, \mathcal{F}) = W_{12} \oplus W_{13} \oplus W_{23} \oplus W_{12}^* \oplus W_{13}^* \oplus W_{23}^* = \mathbb{C}^{12}.$$

Choosing coordinates  $x_{ij}^0, x_{ij}^1$  in  $W_{ij}$  such that  $\tau(x_{ij}^0) = x_{ij}^1$ , let  $y_{ij}^0, y_{ij}^1$  be their duals in  $W_{ij}^*$ .

$\text{Aut}(\mathcal{F}) = \mathbb{C}^{*3}$  by the stability of  $\mathcal{F}_i$ , hence  $G := P\text{Aut}(\mathcal{F}) = \mathbb{C}^{*2}$ , and, setting  $\epsilon_1 := \lambda_1/\lambda_2, \epsilon_2 := \lambda_2/\lambda_3$  for  $(\lambda_1, \lambda_2, \lambda_3) \in \text{Aut}(\mathcal{F})$ , its action on  $\text{Ext}_S^1(\mathcal{F}, \mathcal{F})$  is

$$(\epsilon_1, \epsilon_2) \cdot (x_{ij}^k, y_{ij}^k) = (\epsilon_1 x_{12}^k, \epsilon_1 \epsilon_2 x_{13}^k, \epsilon_2 x_{23}^k, \epsilon_1^{-1} y_{12}^k, \epsilon_1^{-1} \epsilon_2^{-1} y_{13}^k, \epsilon_2^{-1} y_{23}^k).$$

The algebra of invariants of the action of  $G$  on  $\mathbb{P}(\text{Ext}_S^1(\mathcal{F}, \mathcal{F}))$  is generated by the 12 quadratic monomials

$$u_{ij}^{kl} := x_{ij}^k y_{ij}^l \quad i < j,$$

and by the 16 cubic monomials

$$v^{klm} := x_{13}^k y_{12}^l y_{23}^m, \quad w^{klm} := y_{13}^k x_{12}^l x_{23}^m.$$

Its generating relations are the 3 equations between  $u_{ij}^{kl}$

$$u_{ij}^{00} u_{ij}^{11} = u_{ij}^{01} u_{ij}^{10},$$

the 18 equations between  $v^{ijk}, w^{ijk}$

$$\begin{aligned} v^{klm} v^{k'l'm'} &= v^{k'lm} v^{kl'm'} = v^{kl'm} v^{k'l'm'} = v^{klm'} v^{k'l'm}, \\ w^{klm} w^{k'l'm'} &= w^{k'lm} w^{kl'm'} = w^{kl'm} w^{k'l'm'} = w^{klm'} w^{k'l'm}, \end{aligned}$$

the 64 cubic equations

$$v^{klm} w^{k'l'm'} = u_{13}^{kk'} u_{12}^{ll'} u_{23}^{mm'}.$$

2) With the above choice of coordinates  $x_{ij}^k$  and  $y_{ij}^k$ , the action of  $j$  is  $x_{ij}^k \leftrightarrow y_{ij}^k$ , so that

$$\eta^*(x_{ij}^k, y_{ij}^k) = (y_{12}^1, y_{12}^0, y_{13}^1, y_{13}^0, y_{23}^1, y_{23}^0, x_{12}^1, x_{12}^0, x_{13}^1, x_{13}^0, x_{23}^1, x_{23}^0).$$

Its fixed locus is then

$$y_{12}^1 = x_{12}^0, y_{12}^0 = x_{12}^1, y_{13}^1 = x_{13}^0, y_{13}^0 = x_{13}^1, y_{23}^1 = x_{23}^0, y_{23}^0 = x_{23}^1.$$

Since

$$\eta^*((\epsilon_1, \epsilon_2) \cdot (x_{ij}^k, y_{ij}^k)) = \left( \frac{1}{\epsilon_1}, \frac{1}{\epsilon_2} \right) \eta^*(x_{ij}^k, y_{ij}^k),$$

$\eta^*$  induces a well defined involution on  $Ext_S^1(\mathcal{F}, \mathcal{F})//G$ . Again,  $\eta^*$  is not  $G$ -invariant, so its fixed locus on the quotient cannot be expressed as quotient of its fixed locus. But we can describe it using the invariant coordinates  $u_{ij}^{kl}, v^{klm}, w^{klm}$ : the fixed locus of  $\eta^*$  is

$$\begin{aligned} u_{ij}^{00} &= u_{ij}^{11}, \\ v^{klm} &= w^{1-k, 1-l, 1-m}. \end{aligned}$$

The generators of the function algebra of  $(Ext_S^1(\mathcal{F}, \mathcal{F})//G)^{\eta^*}$  are the 9 coordinate functions  $u_{ij}^{00}, u_{ij}^{01}, u_{ij}^{10}$  and the 8 coordinate functions  $v^{klm}$ , while the relations are the 3 equations

$$(u_{ij}^{00})^2 = u_{ij}^{01} u_{ij}^{10},$$

and the 36 equations

$$v^{klm} v^{k'l'm'} = u_{13}^{k, 1-k'} u_{12}^{l, 1-l'} u_{23}^{m, 1-m'},$$

since the set of 18 equations described above follows from these ones.

Now we easily see that these equations describe the quotient  $\mathbb{C}^6/\mathbb{Z}_2 \oplus \mathbb{Z}_2$ . Indeed choosing coordinates  $r_1^0, r_1^1, r_2^0, r_2^1, r_3^0, r_3^1$ , in which the action is given by  $\mathbb{Z}_2 \oplus \mathbb{Z}_2 = \langle (1, 1, -1, -1, -1, -1), (-1, -1, 1, 1, -1, -1) \rangle$ , the algebra of invariant functions is generated by the 9 quadratic monomials

$$s_i^{jk} := r_i^j r_i^k$$

and by the 8 cubic monomials

$$t^{ijk} := s_1^i s_2^j s_3^k,$$

with the 3 quadratic relations

$$s_i^{01} = s_i^{00} s_i^{11},$$

and the 36 cubic ones

$$t^{ijk} t^{i'j'k'} = s_1^{ii'} s_2^{jj'} s_3^{kk'}.$$

Up to the choice of notation, the equations are the same as above.

Hence  $C_{[\mathcal{F}]}(\mathcal{P}) = (Ext_S^1(\mathcal{F}, \mathcal{F})//G)^{\eta^*} = \mathbb{C}^6/\mathbb{Z}_2 \oplus \mathbb{Z}_2$  by dimension reasons.  $\square$



**Corollary 2.14.** *The singular locus of  $\mathcal{P}$  coincides with the locus of  $\eta$ -invariant  $C$ -strictly semistable sheaves. Moreover, there are only quotient singularities, so  $\mathcal{P}$  is an orbifold.*

*Remark 2.15.* It remains an open question if  $\mathcal{P}$  can be expressed as a quotient of another manifold by a regular finite group action. In the following section, we will express it as a quotient of  $S^{[3]}$  by a rational involution.

### 3 Simple connectedness and irreducibility of $\mathcal{P}$

To study the fundamental group and the  $H^{(2,0)}$  of  $\mathcal{P}$ , we describe a birational model of  $\mathcal{P}$  quotienting  $S^{[3]}$  by an involution.

To define it, we first observe that given a generic  $\xi \in S^{[3]}$ ,  $C_\xi := \langle \xi, p_0 \rangle \cap S$  is a generic element of  $|\mathcal{O}_S(1)|^\tau$  and  $C'_\xi := \langle \phi(\xi) \rangle \cap X$  is a generic member of  $|\mathcal{O}_X(1)|$ . Indeed, as remarked before, a  $\tau$ -invariant hyperplane contains  $p_0$ , so it is given by  $p_0$  and 3 other points.

Hence generically  $\xi \in J^3(C_\xi)$ , and  $\xi - \tau(\xi) \in P(C_\xi/C'_\xi)$ .

We thus obtain the natural map

$$\begin{aligned} \psi : S^{[3]} &\dashrightarrow \mathcal{P} \\ \xi = \{p_1, p_2, p_3\} &\mapsto \sum_{i=1}^3 (p_i - \tau(p_i)). \end{aligned}$$

Its indeterminacy locus is generically given by  $\xi$  such that  $\dim \langle \xi \rangle < 3$ , i.e.  $\langle \xi \rangle$  is a line (or a point). A line meeting  $S$  in 3 points, meets also  $Y_2$  in 3 points, so it lies on  $Y_2$ . Vice versa, a line on  $Y_2$ , clearly meets  $Y_3$  in 3 points, so it also meets  $S$  in these 3 points. Hence  $\text{Indet}(\psi) = \{\text{lines in } Y_2\} = \mathbb{P}^3$ .

Let us consider the natural rational involution

$$\begin{aligned} \iota_0 : S^{[3]} &\dashrightarrow S^{[3]} \\ \xi &\mapsto (\langle \xi \rangle \cap S) - \xi. \end{aligned}$$

It is antisymplectic as proven in [O4], Proposition 4.1. Again  $\text{Indet}(\iota_0) = \{\text{lines in } Y_2\}$ . If we consider a generic  $\xi \in \text{Indet}(\iota_0)$ , then  $\langle \xi, p_0 \rangle \cong \mathbb{P}^2$  meets  $S$  in 6 points, respectively  $\xi$  and  $\tau(\xi)$ , and  $\tau(\xi) \in \text{Indet}(\iota_0)$ . So we can extend  $\iota$  to an involution on the blowup  $Bl(S^{[3]})$  of  $S^{[3]}$  along  $\text{Indet}(\psi)$ :

$$\iota_1 : Bl(S^{[3]}) \rightarrow Bl(S^{[3]}).$$

$\tau$  induces a natural involution on  $S^{[3]}$  and on  $Bl(S^{[3]})$ , which we denote again by  $\tau$ . Since it comes from a linear involution, it commutes with  $\iota_0$  and  $\iota_1$ . Setting  $\iota_2 := \iota_0 \circ \tau$ , we get an involution on  $Bl(S^{[3]})$ , which comes from a rational symplectic involution of  $S^{[3]}$ .

**Lemma 3.1.**  $\psi$  is a rational double cover with involution  $\iota_2$ , hence  $M := Bl(S^{[3]})/\iota_2$  is birational to  $\mathcal{P}$ .

*Proof.* Let  $\xi = \{p_1, p_2, p_3\}$  be generic. We want to determine all the divisors  $\xi' = \{p'_1, p'_2, p'_3\}$  on  $C_\xi$  such that  $\xi - \tau(\xi) \sim \xi' - \tau(\xi')$ . Equivalently, setting  $\delta := \xi + \tau(\xi')$ , we want to determine the solutions of  $\delta \sim \tau(\delta)$  for  $\xi$  generic.

If  $\delta = \tau(\delta)$ , then  $\delta$  is  $\tau$ -invariant and, modulo the permutations of  $\xi$  and of  $\xi'$ , we have only 4 possibilities:

- a)  $p'_i = \tau(p_i)$ ,  $i = 1, 2, 3$ , then  $2\xi \sim 2\tau(\xi)$ , hence  $\xi$  is non-generic.
- b)  $p'_1 = \tau(p_1)$ ,  $p'_2 = \tau(p_2)$ ,  $p'_3 = p_3$ , then  $2(p_1 + p_2) \sim 2(\tau(p_1) + \tau(p_2))$ , hence  $\xi$  is non-generic.
- c)  $p'_1 = \tau(p_1)$ ,  $p'_2 = p_2$ ,  $p'_3 = p_3$ , then  $2p_1 \sim 2\tau(p_1)$ , hence  $\xi$  is non-generic.

If  $\delta \neq \tau(\delta)$ , then  $\dim |\delta| > 0$ . By Riemann-Roch theorem we have  $\dim |\delta| = 3 + \dim |K_{C_\xi} - \delta|$ , with  $\deg K_{C_\xi} = \deg \delta = 6$ .

There are three subcases:

d)  $K_{C_\xi} \sim \delta$ , so  $\langle \delta \rangle$  is a plane in  $\langle C_\xi \rangle \cong \mathbb{P}^3$ , and  $|\delta| = \mathbb{P}^{3*}$ . Then  $\tau(\xi')$  is uniquely determined as  $\langle \xi \rangle \cap C_\xi - \xi$ . So the unique nontrivial solution is  $\iota_2(\xi)$ .

e)  $K_{C_\xi} \neq \delta$  and  $|\delta|$  is base point free. Then none of the possible 5-uples of points of  $\delta$  lies on a plane. Now  $|\mathcal{O}_{\mathbb{P}^3}(2)| = \mathbb{P}^9$ , and  $|\mathcal{O}_{C_\xi}(2)| = |2H_{C_\xi}| \cong \mathbb{P}^8$ , since  $C_\xi \subset \langle \xi, p_0 \rangle \cap Y_2$ . So there exist 6 points  $\bar{\delta}$  on  $C_\xi$  such that  $|\delta|$  consists of the residual intersection  $(q \cap C_\xi) - \bar{\delta}$ , where  $q \in |2H_{C_\xi} - \bar{\delta}| = \mathbb{P}^2$ . Moreover,  $\tau$  acts linearly on  $\langle \xi \rangle$ , so  $q \in |2H_{C_\xi} - \bar{\delta}|$  if and only if  $\tau(q) \in |2H_{C_\xi} - \tau(\bar{\delta})|$ . As  $\delta \sim \tau(\delta)$ , the two families coincide. We deduce that  $\bar{\delta}$  is  $\tau$ -invariant, and hence every quadric in  $|2H_{C_\xi} - \bar{\delta}|$  is  $\tau$ -invariant. Thus  $\xi = \xi'$ , i.e.  $\delta = \tau(\delta)$ , absurd.

f)  $K_{C_\xi} \neq \delta$ , and  $|\delta| = \mathbb{P}^2$  has a base point. Then 5 points of  $\delta$  span a plane  $\Pi$ ; assume they are  $p_1, p_2, p_3, \tau(p'_1), \tau(p'_2)$ . Setting  $\bar{p}$  the remaining intersection point of  $\Pi$  with  $C_\xi$ ,  $|\delta|$  is clearly given by  $|H_{C_\xi} - \bar{p}| = \mathbb{P}^2$ . As  $\delta \sim \tau(\delta)$ ,  $\bar{p}$  is  $\tau$ -invariant. So  $\xi$  spans a plane passing through one of the six  $\tau$ -invariant points of  $C_\xi$ , hence  $\xi$  is non-generic.

We conclude that the generic fiber of  $\psi$  consists of only two points, interchanged by  $\iota_2$ .  $\square$

**Corollary 3.2.**  $h^{(2,0)}(\mathcal{P}) = h^{(2,0)}(M) = 1$ .

**Lemma 3.3.**  $\text{Fix}(\iota_2)$  is the union of two smooth irreducible 4-folds and 120 isolated points.

*Proof.* Obviously, the fixed locus of a biregular involution on a smooth variety is also smooth.

For a generic  $\iota_2$ -invariant  $\xi$ , the plane  $\langle \xi \rangle$  is  $\tau$ -invariant, because the planes are  $\iota_1$ -invariant. Recalling that  $\text{Fix}(\tau) = H_4 \cup p_0$ , either  $\langle \xi \rangle \subset H_4$  or  $p_0 \in \langle \xi \rangle$  (indeed if  $\exists p \in \langle \xi \rangle - H_4$ , then  $p_0 \in \langle \xi \rangle$ ).

In the first case  $\xi \subset S \cap H_4$  is  $\tau$ -invariant, so  $\iota_1(\xi) = \xi$ . Then  $\langle \xi \rangle$  is totally tangent to the curve  $S \cap H_4$ . This imposes 3 conditions in  $\mathbb{P}^3$ , hence

we expect a finite number of such  $\xi'$ s. These correspond to the odd theta characteristics of the curve, which are exactly  $2^3(2^4 - 1) = 120$ .

In the second case, we obtain the remaining part of  $\text{Fix}(\iota_2)$  as

$$\Sigma := \{\xi \in S^{[3]} : p_0 \in \langle \xi \rangle\}.$$

To describe it, we consider the natural map  $Bl(S^{[3]}) \rightarrow \mathbb{G}(2, 4)$ , implicitly described before. It is a  $\binom{6}{3} = 20$ -to-1 covering. The image of  $\Sigma$  is clearly  $\{\Pi \text{ plane} \subset \mathbb{P}^4 : p_0 \in \Pi\} = \sigma_{1,1} \cong \mathbb{G}(1, 3)$ , so  $\Sigma$  is a 20:1 covering of a smooth quadric of  $\mathbb{P}^5$ . Since  $\langle \xi \rangle$  is  $\tau$ -invariant,  $\langle \xi \rangle \cap S = \{\xi, \tau(\xi)\}$ . The 12 triples  $\{p_i, \tau(p_i), p_j\}$  and  $\{p_i, \tau(p_i), \tau(p_j)\}$  sweep a 4-fold  $\Sigma_1 \subset S^{[3]}$  which is a double covering of  $X^{[2]}$ . The other 8 triples sweep  $\Sigma_2$ , an 8-sheeted covering of  $\sigma_{1,1}$ . So  $\Sigma$  has 2 disjoint irreducible components,  $\Sigma_1$  and  $\Sigma_2$ .  $\square$

*Remark 3.4.* Considering  $\iota_2$  as a rational involution on  $S^{[3]}$ , it has the same fixed locus, because a line on  $Y_2$  does not lie in  $H_4$  and does not pass through  $P_0$ .

**Proposition 3.5.**  *$\mathcal{P}$  and  $M := Bl(S^{[3]})/\iota_2$  are simply connected.*

*Proof.* Since there are fixed points,  $M$  has the same fundamental group as  $Bl(S^{[3]})$ , which has the same fundamental group as  $S^{[3]}$  because it is a blowup along a smooth locus.

$M$  has singularities of type  $\mathbb{C}^4 \times (\mathbb{C}^2 / -1)$  on  $\Sigma_i$  and of type  $\mathbb{C}^6 / -1$  at the isolated points - see the invariant part of the action of  $\eta_*$  of  $T_P(S^{[3]})$ .

All the singularities of  $\mathcal{P}$  can be resolved by blowups by Hironaka theorem, so it is simply connected.  $\square$

*Remark 3.6.* Since the singularities of  $M$  and of  $\mathcal{P}$  are different, they are only birational. It is interesting to determine explicitly a birational transformation between them.

## 4 Euler characteristic of $\mathcal{P}$

To calculate the Euler characteristic of  $\mathcal{P}$ , we can use the fibration structure. Since the Euler characteristic is additive, i.e.  $\chi(X) = \chi(U) + \chi(X - U)$  for any open  $U \subset X$ , and multiplicative for trivial fibrations, i.e.  $\chi(X \times Y) = \chi(X) \cdot \chi(Y)$ , we can stratify  $|\mathcal{O}_S(1)|^\tau = |\mathcal{O}_X(1)| = \mathbb{P}^{3*}$  depending on the fibers. Since  $\chi = 0$  for any smooth abelian variety, it is enough to study the locus of singular fibers. They come from the singular curves of the linear system. This subset of  $\mathbb{P}^{3*}$  is called discriminant of the fibration. We denote it by  $\Delta$ . In the following lemma we describe it.

**Lemma 4.1.** *The discriminant  $\Delta \subset \mathbb{P}^{3*}$  consists of two irreducible components: the dual  $X^*$  of the cubic surface, of degree 12, and the dual  $B^*$  of the branch locus of  $\phi$ , of degree 18.*

*Proof.* Clearly,  $C$  is singular if and only if  $C'$  is singular or  $C'$  is tangent to  $B$ .

The degrees of  $X^*$  and  $B^*$  can be easily determined using Schubert calculus in  $\mathbb{P}^{3*}$ :

$$\deg X^* = X^* \cap \sigma_{1,0,0}^2 = X^* \cap \sigma_{1,1,0} = \{H : l \subset H, H \text{ tangent to } X\},$$

$$\deg B^* = \{H : l \subset H, H \text{ tangent to } B\},$$

with  $l$  a generic line in  $\mathbb{P}^3$ . Denoting  $P, Q, R$  the intersection points of  $X$  and  $l$ , we have a natural map  $f : Bl_{P,Q,R}(X) \rightarrow \sigma_{1,1,0} \cong \mathbb{P}^1$  such that  $f(p) = \langle p, l \rangle$ . The degree of  $X^*$ , which is the number of planes tangent to  $X$  and passing through  $l$ , corresponds to the number  $N$  of singular fibers of  $f$ . Using the good properties of the Euler characteristic, we get

$$\chi(Bl_{P,Q,R}(X)) = \chi(\mathbb{P}^1 - N \text{ pts})\chi(\text{smooth fiber}) + \chi(N \text{ pts})\chi(\text{singular fiber}),$$

i.e.  $N = 12$ , because  $\chi(Bl_{P,Q,R}(X)) = \chi(Bl_{9\text{pts}}(\mathbb{P}^2)) = \chi(\mathbb{P}^2) + 9 = 12$  and a smooth fiber has  $\chi$  equal to zero (it is an elliptic curve), while a singular fiber has  $\chi = 1$  (it is a nodal plane cubic).

To determine  $\deg B^*$ , we can consider the 6:1 cover  $g : B \rightarrow \sigma_{1,1,0} \cong \mathbb{P}^1$  such that  $g(p) = \langle p, l \rangle$ . The degree corresponds to the degree of the branch locus, which is 18 by Riemann-Hurwitz theorem.  $\square$

*Remark 4.2.* The degree of the discriminant locus of  $\mathcal{P}$  is 30. For irreducible symplectic 6-folds obtained as universal Jacobians of a linear system of curves on a K3 surface, the so called Beauville-Mukai integrable systems, the degree is 36. General results on the degree of a Lagrangian fibration with Jacobians of integral curves as fibers have been obtained by Sawon in [S1].

We focus on the natural stratification of  $\Delta$ , which will permit us to determine  $\chi(\mathcal{P})$ .

**Proposition 4.3.** *For a generic  $S$ ,  $\Delta$  admits a natural stratification in singular loci (with several irreducible components), corresponding to all the possible singular members of  $|\mathcal{O}_S(1)|^\tau$ , as described in the following:*

Dimension 2

- a)  $C$  has a simple  $\tau$ -invariant node, if  $\langle C' \rangle \in B^* - \text{Sing}(B^* \cup X^*)$  (i.e.  $C'$  is tangent to  $B$ );
- b)  $C$  has two simple nodes, interchanged by  $\tau$ , if  $\langle C' \rangle \in X^* - \text{Sing}(B^* \cup X^*)$  (i.e.  $C'$  has a simple node);

Dimension 1

- c)  $C$  has two simple  $\tau$ -invariant nodes, if  $\langle C' \rangle$  lies in the complement of  $\text{Sing}(\text{Sing}(B^* \cup X^*))$  inside the irreducible component of  $\text{Sing}(B^*)$  corresponding to  $C'$  bitangent to  $B$ ;
- d)  $C$  has one cusp, if  $\langle C' \rangle$  lies in the complement of  $\text{Sing}(\text{Sing}(B^* \cup X^*))$  inside the irreducible component of  $\text{Sing}(B^*)$  corresponding to  $C'$  with a triple tangency point on  $B$ ;
- e)  $C$  has three simple nodes, one fixed by  $\tau$  and the others interchanged by  $\tau$ , if  $\langle C' \rangle$  lies in the complement of  $\text{Sing}(\text{Sing}(B^* \cup X^*))$  inside the irreducible component of  $B^* \cap X^*$  corresponding to  $C'$  tangent to  $B$  and having a simple node;
- f)  $C$  has a tacnode, if  $\langle C' \rangle$  lies in the complement of  $\text{Sing}(\text{Sing}(B^* \cup X^*))$  inside the irreducible component of  $B^* \cap X^*$  corresponding to  $C'$  with a simple node on  $B$ , in other words  $C'$  is cut out by a plane tangent to  $X$  at a point of  $B$ ;
- g)  $C$  has two simple cusps, interchanged by  $\tau$ , if  $\langle C' \rangle$  lies in the complement of  $\text{Sing}(\text{Sing}(B^* \cup X^*))$  inside the irreducible component of  $\text{Sing}(X^*)$  corresponding to  $C'$  with a cusp;
- h)  $C$  has two irreducible components meeting in four points, interchanged in pairs by  $\tau$ , if  $\langle C' \rangle$  lies in the complement of  $\text{Sing}(\text{Sing}(B^* \cup X^*))$  inside the irreducible component of  $\text{Sing}(X^*)$  corresponding to  $C'$  reducible plane cubic decomposing into a conic and a line;

Dimension 0

- i)  $C$  has a cusp and a simple node, if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(\text{Sing}(B^*))$  corresponding to  $C'$  with a triple tangency point on  $B$  and another simple tangency point;  $c) \cap a), d) \cap a)$
- j)  $C$  has a tacnode, if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(\text{Sing}(B^*))$  corresponding to  $C'$  with a quadruple tangency point on  $B$ ;  $d) \cap a)$
- k)  $C$  has three simple  $\tau$ -invariant nodes, if  $\langle C' \rangle$  is one of the 120 points of  $\text{Sing}(\text{Sing}(B^*))$  corresponding to  $C'$  with a tritangent to  $B$ ;  $c) \cap a)$
- l)  $C$  has two simple cusps interchanged by  $\tau$  and a simple  $\tau$ -invariant node, if  $C'$  is one of the points of  $B^* \cap \text{Sing}(X^*)$  corresponding to  $C'$  tangent to  $B$  with a simple cusp;  $g) \cap a), e) \cap b)$
- m)  $C$  has an  $A_5$  singularity, if  $\langle C' \rangle$  is one of the points of  $B^* \cap \text{Sing}(X^*)$  corresponding to  $C'$  with a simple cusp on  $B$ ;  $g) \cap a), e) \cap b), f) \cap b)$
- n)  $C$  has two irreducible components meeting in pairs in two points interchanged by  $\tau$  and a simple node only on one of the two components, if

- $\langle C' \rangle$  is one of the points of  $B^* \cap \text{Sing}(X^*)$  corresponding to  $C'$  reducible plane cubic consisting of a line and a conic tangent to  $B$ ;  $h) \cap a), e) \cap b)$
- o)  $C$  has two  $\tau$ -invariant simple nodes and two simple nodes interchanged by  $\tau$ , if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(B^*) \cap X^*$  corresponding to  $C'$  bitangent to  $B$  and having a simple node;  $e) \cap a), c) \cap b)$
  - p)  $C$  has a tacnode and a simple node, if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(B^*) \cap X^*$  corresponding to  $C'$  tangent to  $B$  and having a singular point on  $B$ ;  $e) \cap a), f) \cap a), c) \cap b)$
  - q)  $C$  has a  $D_4$  singularity, if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(B^*) \cap X^*$  corresponding to  $C'$  with a singular tangent point on  $B$ ;  $f) \cap a), d) \cap b)$
  - r)  $C$  has two simple nodes interchanged by  $\tau$  and a simple  $\tau$ -invariant cusp, if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(B^*) \cap X^*$  corresponding to  $C'$  with a triple tangency point on  $B$  and a singular point;  $e) \cap a), d) \cap b)$
  - s)  $C$  has three irreducible components meeting in pairs in two points interchanged by  $\tau$ , if  $\langle C' \rangle$  is one of the points of  $\text{Sing}(\text{Sing}(X^*))$  (i.e. a reducible plane cubic given by three lines).  $h) \cap b)$

*Proof.* As in the previous description of the discriminant locus, to describe the singularities of  $C$  it is enough to look at  $C'$ . If  $C'$  has a simple node/cusp outside  $B$ , then  $C$  inherits two nodes/cusps interchanged by  $\tau$ . If  $C'$  has a double/triple/quadruple tangency point with  $B$ , then  $C$  inherits a  $\tau$ -invariant simple node/cusp/tacnode. If  $C'$  has a simple node on  $B$ , then  $C$  has a tacnode, because locally  $C'$  has equations  $u^2 + v^2 = 0$  hence  $C$  is  $t^2 = u, u^2 + v^2 = 0$ , i.e.  $t^4 + v^2 = 0$ . If  $C'$  has a simple node on  $B$  and  $B$  is tangent to one of the two branches of the curve through it, then  $C$  has a  $D_4$ -singularity, because locally  $C'$  has equations  $uv + v^3 = 0$ , hence  $C$  is  $t^2 = u, uv + v^3 = 0$ , i.e.  $(t^2 + v^2)v = 0$ . If  $C'$  has a simple cusp on  $B$ , then  $C$  has an  $A_5$ -singularity, because locally  $C'$  has equations  $u^3 + v^2 = 0$  hence  $C$  is  $t^2 = u, u^3 + v^2 = 0$ , i.e.  $t^6 + v^2 = 0$ . □

Continuing the calculation of  $\chi(\mathcal{P})$ , we denote by  $\Pi_\bullet$  the locus of points such that the condition  $\bullet$  of the previous proposition holds, and by  $\bar{P}_\bullet$  the fiber of a point of  $\Pi_\bullet$  (i.e. the compactified Prym variety of a curve from  $\Pi_\bullet$ ). As explained above, we get

$$\chi(\mathcal{P}) = \chi(\Pi_a)\chi(\bar{P}_a) + \dots + \chi(\Pi_s)\chi(\bar{P}_s).$$

To calculate  $\chi(\bar{P}_\bullet)$ , we follow a description of  $\bar{P}_\bullet$  used in [MT], based on a description of  $\bar{J}(C)$  by Cook in [C]. We recall that  $\bar{J}(C)$  admits a stratification in smooth strata whose codimension is equal to the index

$i(\mathcal{F})$  of the sheaves  $\mathcal{F}$  represented by points of these strata. The normalization map  $\nu : \tilde{C} \rightarrow C$  factorizes through a partial normalization  $\nu' : \tilde{C} \rightarrow C$  such that  $\nu'^*(\mathcal{F})/(torsion)$  is invertible, and  $i(\mathcal{F})$  is the minimum of  $length(\nu'_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C)$ . When  $C$  is integral,  $\bar{J}(C)$  is irreducible by [AIK], and the index takes values between 0 and  $\delta(C) = length(\nu'_*(\mathcal{O}_{\tilde{C}})/\mathcal{O}_C) = p_a(C) - g(C)$ . Each stratum can be described as an extension of  $J(\tilde{C})$  by an algebraic group. Let  $J_i(C)$  be the union of the strata of codimension  $i$  ( $0 \leq i \leq 4$ ). So  $J_0(C) = J(C)$ . The map  $\mathcal{F} \mapsto \nu^*(\mathcal{F})/(torsion)$ , restricted to  $J_i(C)$ , gives a morphism  $v_i : J_i(C) \rightarrow Pic^{-i}(\tilde{C})$ .

We denote by  $P_i$  the stratum  $J_i(C) \cap \bar{P}$  induced on  $\bar{P}$ . So  $P_0 = P(C/C')$ , an algebraic group of dimension 4. Moreover,  $\tau$  extends to an involution on  $\tilde{C}$  corresponding to the double cover  $\tilde{C} \rightarrow \tilde{C}'$ , with  $\tilde{C}'$  normalization of  $C'$ . Each stratum can be obtained as an extension of  $P(\tilde{C}/\tilde{C}')$  with an algebraic group.

As explained in [B3], Proposition 2.2,  $\chi(P_\bullet)$  corresponds to the number of 0-dimensional strata of  $P_\bullet$ . Hence it is nonzero only in the cases  $k), n), o), s)$ . Hence it is enough to determine the cardinality of  $\Pi_\bullet$  and the 0-dimensional strata of  $P_\bullet$ .

**Lemma 4.4.**  $\chi(\mathcal{P}) = 2610$ .

*Proof.*  $k)$  The number of tritangents to  $B$  corresponds to the number of odd theta characteristics. Hence we get  $2^3(2^4 - 1) = 120$  points.

We need to determine the zero-dimensional strata of  $\bar{P}$ , which is  $P_3$ , because  $C$  is irreducible and  $\delta(C) = 3$ . First, we observe that  $P(\tilde{C}/C')$  is given by two points. Indeed  $C$  has three  $\tau$ -invariant nodes,  $p_1, p_2, p_3$ . By Riemann-Hurwitz, the induced  $\tau$  on  $\tilde{C}$  is base-point-free, so  $\tau(p'_i) = p''_i$ , with  $\nu^{-1}(p_i) = \{p'_i, p''_i\}$ , and  $\tau$  is a translation by a 2-torsion point  $q = [p'_1 - p''_1] = [p'_1 - p''_2] = [p'_3 - p''_3] \in J(\tilde{C}) = \tilde{C}$ . So  $\eta$  has 4 fixed points on  $\tilde{C}$  (the four solutions of  $2p = q$ ) and  $P(\tilde{C}/C')$  consists of two points.

Following Cook [C], the elements of  $J_3(C)$  are of the form  $\nu_*(\mathcal{L})$ , with  $\mathcal{L} \in Pic^{-3}(\tilde{C})$ . To determine  $P_3$ , we need to describe the action of  $\eta$  on  $J_3(C)$ :

$$j(\nu_*(\mathcal{L})) = \nu_*((\mathcal{L}^{-1})(-p'_1 - p''_1 - p'_2 - p''_2 - p'_3 - p''_3)),$$

$$\tau(\nu_*(\mathcal{L})) = \nu_*(\tau(\mathcal{L})).$$

Hence  $\nu_*(\mathcal{L}) \in P_3$  if and only if  $\mathcal{L} \in P(\tilde{C}/C')$ , and  $P_3$  consists of two points.

$n)$  The number of reducible curves given by a conic tangent to  $B$  and a line, corresponds to the intersection number of  $h$ ) and  $a$ ).  $B^*$  has degree 18 as shown before, while the degree of the reducible curves



given by a conic and a line is 27, because there are 27 lines on a cubic. So we get  $18 \cdot 27 = 486$  points.

The zero-dimensional stratum of  $\bar{P}$  is  $P_5$ , because  $C$  has four simple nodes.  $P(\tilde{C}/\tilde{C}')$  is a point, because  $\tilde{C}$  and  $\tilde{C}'$  are rational curves. The elements of  $J_5(C)$  are of the form  $\nu_*(\mathcal{O}_{\tilde{C}}(d-1)) \oplus \mathcal{O}_{\tilde{C}}(-d)$ , for  $d$  satisfying semistability conditions, i.e.  $d = 0, \pm 1, \pm 2$  ( $\pm 1$  represent the same  $S$ -equivalence class, and also  $\pm 2$ ).  $P_5$  thus consists of three points.

- o) To determine the number of nodal curves bitangent to  $B$ , we calculate it indirectly, determining the degree of the curve (case  $c$ ) of bitangents to  $B$  and the number (case  $p$ ) of curves tangent to  $B$  and having a singular point on  $B$ . Indeed,  $b) \cap c) = 2p) + o)$ , since  $b)$  and  $c)$  meet transversely and  $p)$  has intersection multiplicity 2 because the bitangents of  $b)$  can acquire a node in one of the two tangency points.

The degree of  $c)$  can be obtained considering a projection of  $B$  onto a plane from a generic fixed point: the number of bitangents of  $B$  corresponds to the number of bitangents of the image  $B'$ , which is a plane curve with the same geometric genus and degree, i.e.  $g = 4$  and  $d = 6$ . Since the arithmetic genus of a plane sextic is 10,  $B'$  has 6 simple nodes. By Plücker formulas, we have

$$g = (d^* - 1)(d^* - 2)/2 - b - f, \quad d = d^*(d^* - 1) - 2b - 3f,$$

where  $d^*$  is the degree of the dual curve of  $B'$ ,  $b$  is the number of bitangents,  $f$  the number of flexes. So we need to determine  $d^*$ . Again by Plücker formulas

$$d^* = d(d-1) - 2\delta - 3\kappa,$$

where  $\delta$  is the number of simple nodes and  $\kappa$  the number of cusps. Hence  $d^* = 18$  and  $b = 90$ , so  $c)$  has degree 90.

The degree of  $p)$  can be obtained considering the curve of the case  $f)$ . From it we can define

$$D := \{\Pi \cap B - \{p\} : \Pi \text{ tangent to } X \text{ at } p\}_{p \in B}.$$

It is a 4:1 cover of  $B$  with branching  $p)$ . By Riemann-Hurwitz, to calculate  $p)$ , it is enough to determine the genus of  $D$ .  $D$  can be seen as a subvariety of  $B \times B \subset \mathbb{P}^3 \times \mathbb{P}^3$ : the equation  $\sum x_i \partial_i F(\underline{p}) = 0$ , with  $((x_i), (\underline{p})) \in B \times B$ , gives  $D + 2\Delta_B$ . Setting  $f_1 := \bar{B} \times pt$ ,  $f_2 := pt \times \bar{B}$ , we get that  $D \sim 12f_1 + 6f_2 - 2\Delta_B$  numerically, hence  $K_D \sim (K_{B \times B} + D)D \sim (18f_1 + 12f_2 - 2\Delta_B)(12f_1 + 6f_2 - 2\Delta_B)$  and using the intersection relations  $\Delta_B \cdot f_1 = \Delta_B \cdot f_2 = 0, f_1^2 = f_2^2 = 0, \Delta_B^2 = \deg \mathcal{N}_B = \deg \mathcal{T}_B = 2 - 2g_B = -6$ , we have  $K_D \sim 132$ . So

$g(D) = 67$ , and by Riemann-Hurwitz the branch locus consists of 108 points.

In conclusion,  $o)$  consists of  $90 \cdot 12 - 2 \cdot 108 = 864$  points. The zero-dimensional stratum of  $\bar{P}$  is  $P_4$ , because  $C$  is irreducible and  $\delta(C) = 4$ .  $P(\tilde{C}/\tilde{C}')$  is a point, because  $\tilde{C}$  and  $\tilde{C}'$  are rational curves. Similarly to  $k)$ , the elements of  $J_4(C)$  are of the form  $\nu_*(\mathcal{O}_{\tilde{C}}(-4))$ , and  $P_4 = P(\tilde{C}/\tilde{C}')$  is a point.

- $s)$  We have 48 points, which are the intersection points of the orthogonal lines to the 27 lines on  $X$ .

Collecting the previous calculation, we obtain

$$\chi(\mathcal{P}) = 120 \cdot 2 + 486 \cdot 3 + 864 \cdot 1 + 48 \cdot 1 = 2610.$$

□

*Remark 4.5.* Many computations of this proof are similar to those of Proposition 4.3 of [MT]. In particular, we remark that at point  $iv)$  there is a mistake: indeed  $P_2$  consists of 2 points, not 4 (and analogously in  $P_1$  and  $P_0$  there are half of the copies of  $\mathbb{C}^*$  and of  $\mathbb{C}^* \times \mathbb{C}^*$ ). This follows from the same considerations as in the previous proof for the item  $k)$ . For this reason, the computation of the Euler characteristic of the 4-fold in [MT] should be corrected as follows:

$$\chi(\mathcal{P}) = 28 \cdot 2 + 128 \cdot 1 + 28 \cdot 1 = 212.$$

This computation agrees with the one done by Menet in [Me], Proposition 2.40, where he determines the Euler characteristic of the 4-fold relating it to the quotient of a K3 surface by an involution.

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